Analogy between scale symmetry and relativistic mechanics. I. Lagrangian formalism

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Using only the logarithm of physical quantities, we show that the equivalence of all systems of units is deeply analogous to symmetry by translation in mechanics. Similarly, the equivalence of all systems of units and subunits helps to generalize usual dimensional analysis, in a curious analogy with speed relativity in mechanics. This analogy leads to nontrivial practical applications when applied to random fields, whose moments combine measurements at different scales. [S1063-651X(97)12811-2]

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I. INTRODUCTION

A century ago, Pierre Curie suggested how physicists should use symmetries: ''Construct *a priori* equations, such that they respect invariance laws; then confront them to experiments.'' Our aim is to apply his suggestion to invariance by dilatation.

Since all systems of units are equivalent, laws of physics are invariant under a multiplication of basic units. This constancy leads to classical dimensional analysis [1–4]. Formally, if X is the logarithm of a physical quantity A, and $X_0=\ln(A_0)$ the logarithm of the unit chosen to measure it, $X-X_0$ is (the logarithm of) the result of the measurement: laws of physics are expressed in a form invariant under a translation applied to X_0.

Thus invariance by dilatation is trivially analogous to the invariance by translation we use in real space. Both symmetries have the same status, namely, they are exact as far as laws of physics are concerned, but when it comes to actual objects they are always broken by boundary conditions (lower and upper cutoffs in the case of dilatation symmetry). Pushing this analogy further leads however to nontrivial results, as we now show (Table I).

II. POSTULATE

In fact, since all systems of units are really equivalent, laws of physics must also remain invariant under a multiplication of basic units *and subunits*.

A. Equivalence between systems of units

To measure a physical quantity necessitates a direct comparison with a reference. This is possible only if the reference has a scale comparable with the quantity to be measured. You do not use the same apparatus to measure lengths in astrophysics, or in nuclear physics. Converting one scale to another is a difficult metrology problem, exactly as the conversion from one unit to the other. The first comparison of the Earth meridian with a platinum meter (and, a century later, with an atomic wavelength) involved a challenging cascade of multiple comparisons. Secondary references, i.e., subunits or surunits, are as vital as basic units themselves. A unit system, based on a unit A_0, always defines a complete set A_i of subunits, with i a positive or negative integer. They have a significance independent of each other; for instance, astronomical unit, light year and parsec are simply defined in relation to each other, and used for precision measurement, even without knowing precisely their value expressed in meters. Each measurement setup operates only in a given, finite range, say around the nth subunit A_n.

The resolution of the system, around this scale, is the ratio $K_n=A_n/A_{n-1}$ of two successive subunits. The nth scale is then

$$\ell_n=\ell_0K^n=\ell_0\times K\times\cdots\times K. \tag{1}$$

Here n is an integer number, ℓ_0 the unit chosen for the scales, and K the chosen resolution, meaning that $\ln K$ is the logarithmic increment between two successive subunits. For instance, $K=2$ for a block renormalization, or K infinitely close to 1 for a continuous renormalization; for simplicity we consider only the case $K>1$ (see Ref. [15] in Sec. III B). The physical laws describing the properties of ℓ are expressed using the label n; their formulation should not depend on the choices of ℓ_0 and K.

Thus the log coordinates [5]

$$T=\frac{1}{\ln K}\ln\left(\frac{\ell}{\ell_0}\right),\qquad X=\frac{1}{\ln K'}\ln\left(\frac{A}{A_0}\right) \tag{2}$$

are not only invariant under an arbitrary choice of the origin $(T_0,X_0)=(\ln\ell_0,\ln A_0)$: they are also invariant under an arbitrary change of $(\ln K,\ln K')$, i.e., a variation of the origin with scale, or ''gauge invariance'' taken in its original acception [6]. These symmetries are sometimes called ''global'' and

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TABLE I. Proposed analogy between the present series of articles and relativistic mechanics. Notations are defined as in Ref. [9].

Exponent relativity	Speed relativity
Definitions	
log of the scale T	time t
log of the moment X	position x
unit ℓ_0	origin of time axis t_0
reference field $\mathcal{R}$	origin of space axis x_0
system of units and subunits	system of coordinates
scaling exponent $\dot{X}=dX/dT$	speed $\dot{x}=dx/dt$
power-law reference field	tangent Galilean reference frame
relative variation of reference fields: exponent $V_{\mathcal{R}'\mid\mathcal{R}}$	relative speed of reference frames: $v_{x_0'\mid x_0}$
statistics	kinematics
variation with scale	dynamics
Symmetries	
free choice of the system of subunits	free choice of referential
arbitrary choice of units	arbitrary origin of space
scale-dependent resolution	arbitrary coordinates
scale symmetry of the system	homogeneity of space
lower and upper cutoffs	boundary conditions
single variable transformation	locality of physical laws
no analog symmetry $\ell\rightarrow 1/\ell$	space and/or time symmetry
concavity of intermittency function	causality
Transformation group	
transformations between moments	inertial transformations
characteristic exponent C	characteristic speed c
$\Lambda\geqslant 1$	$\Lambda\equiv 0$
$\Gamma(V)=[1-2\Lambda V/C+(\Lambda^2-1)V^2/C^2]^{-1/2}$	$\gamma(v)=[1-v^2/c^2]^{-1/2}$
invariant exponents C_+,C_-	invariant celerities $+c,-c$
random process	particle trajectories, or field
scale invariant process	free particle or free field
intermittent structures	massless particles

''local'' scale invariances [7]. When expressed in log variables, a scaling law $A\sim\ell^{\zeta}$ with a fixed exponent ζ becomes $X\sim VT$, analog to a trajectory with a fixed velocity. While the former invariance is analogous to translation invariance on (T_0,X_0), the latter invariance is exactly analogous to Galilean invariance, namely, a (Newtonian or Einsteinian [7–11]) change between referentials moving at a constant relative rate.

This implies that laws of physics can express relations between exponents defined as dX/dT (say ''this exponent is twice that other one'') but not single out a privileged value (''this exponent is equal to three''). However, up to now there has been nothing very new in it. Put it another way: by writing $T=\log_K(\ell/\ell_0)$ and $X=\log_{K'}(A/A_0)$, we see that Eq. (2) is nothing more than the freedom to choose the basis of the logarithm. Since it is not crucial, the international system chose $K=K'=10$ consistently, and forgot about it. For deterministic fields, this is in fact a trivial consequence of a well-known property: if a quantity ϕ scales with ℓ, then ϕ^{α} also scales with ℓ^{β} with another exponent [12].

For random fields, $\langle\phi^{\alpha}\rangle$ and $\langle\phi\rangle^{\alpha}$ can be very different, and this property leads to interesting consequences. In fact, successive moments of a random field combine measurements at different scales, so that T and X become coupled variables (in the same way as space and time are coupled in Einsteinian mechanics). Instead of being additive, scaling exponents obey a composition law analog to Lorentz composition for velocities [9,14,7]. This analogy is profound, and exact in most details (Table I). But before turning to nontrivial applications, which will be explored below, let us first speculate on the last consequences of our postulate.

B. Distinctions between systems of units

More generally, the ratio between successive subunits can vary with scale. For instance, foot-pound-hour or centimeter-gram-second is as acceptable as meter-kilogram-second and inch-foot-mile constitutes an acceptable system of subunits. The resolution $K_n=\ell_{n+1}/\ell_n$, which varies with the scale

$$\ell_n=\ell_0\times K_1\times\ldots\times K_n, \tag{3}$$

is now analogous to general relativity, where your coordinate system (and hence your unit, with which you measure the distance between two positions) may depend on the position itself (Appendix A).

Of course, as long as the scale space is flat and not curved, laws will take a much simpler form in a ''Galilean referential'' system, i.e., a system where subunits follow a regular geometrical progression. Nothing constrains $K(\ell)$, but life is easier with K constant. Try to write down Newton's gravitational law under a universal form valid at any scale, using feet and yards. Some unit systems are really more equivalent than others, as George Orwell would say [G. Orwell, *Animal Farm* (Martin Secker & Warburg, London, 1945).

Conversely, if the scale space is curved, there is still a subunit system in which laws of physics take their simpler expression, but it can vary with the scale. Can we imagine such a curvature of the scale space? Curvature which cannot be suppressed only by changing the metrics should be associated with a coupling between different scales. For instance, a wholesaler buying 1000 roses at a cheap price (money to rose conversion factor is low at large scale), retails them at a higher unit price (money to rose conversion factor for only one rose is high). He puts his benefit aside and starts a new cycle. His benefit is the curvature enclosed by his cycle, after parallel transportation in the scale space curved by the nonlinear relation between roses and money.

III. RANDOM FIELDS

Let us now consider a ''process'' ϕ, i.e., a scalar random field, e.g., in isotropic homogeneous turbulence [10]. The variables (T,X) define a space-time with only 1+1 dimensions. In relativistic mechanics, this implies that clocks can be unambiguously synchronized in the whole space: here the log scale T can be univocally defined. One can always construct an inertial referential so that physical laws take a simpler expression; the formalism becomes global instead of local.

We thus concentrate on a formalism with resolutions K and K' which do not depend on scale, which means they are not pertinent anymore [12]. Instead of ϕ, we prefer to deal only with deterministic numbers, and thus use the connection between possible values of a random variable and its moments [14].

The complete formalism is tedious; see Appendix A. With constant resolutions, Eqs. (A1) and (A5) yield simple expressions for log variables:

$$T=\ln\left(\frac{\ell}{\ell_0}\right),$$

$$X(T)=\frac{d\ \ln\langle(\phi_\ell/\mathcal{R}_\ell)^p\rangle}{dp}=\frac{\langle\ln(\phi_\ell/\mathcal{R}_\ell)(\phi_\ell/\mathcal{R}_\ell)^p\rangle}{\langle(\phi_\ell/\mathcal{R}_\ell)^p\rangle}, \tag{4}$$

where ϕ is a random positive physical quantity, and ϕ_ℓ the same field defined at scale ℓ. X thus depends on p and is defined from $p_{\min}$ to $p_{\max}$, characterizing the lower and higher convergent moments of the distribution. As p varies between these limiting values, X takes all the possible values of the logarithm of the random field ϕ_ℓ.

These notations turn particularly convenient; for instance, the variable $\dot{X}=dX/dT$ is the multifractal exponent of the random field [14]. The analogy we develop below reveals their physical significance.

A. Group law for scale transformations

We thus fall back to our previous papers [9,14], to which we refer the reader for more details. Briefly, we defined a similarity transformation analogous to Lorentz transformation, connecting different values of the random fields, at different scales, in different realizations. It is obtained while changing from a first moment n to a new one p [14], or, equivalently, from a first reference field $\mathcal{R}$ to a new one $\mathcal{R}'$ moving with a relative exponent $V_{\mathcal{R}'|\mathcal{R}}$ [9]. Two parameters are necessary: the first one, an exponent characteristic of the physical system, is noted C to stress the analogy to the Lorentz group; the second one, Λ, characterizes the symmetry breaking between large scales and small scales: it breaks the parity $T\to-T$, $X\to-X$ or $V\to-V$.

The matrix $\overline{\overline{S}}(V)$ of similarity transformations is written

$$\begin{pmatrix} CT' \\ X' \end{pmatrix}=\overline{\overline{S}}(V)\begin{pmatrix} CT \\ X \end{pmatrix}=\Gamma(V)\begin{pmatrix} 1-2\Lambda V/C & (\Lambda^2-1)V/C \\ -V/C & 1 \end{pmatrix}\begin{pmatrix} CT \\ X \end{pmatrix}, \tag{5}$$

where we defined

$$\Gamma(V)=\frac{1}{\sqrt{(\Lambda^2-1)V^2/C^2-2\Lambda V/C+1}}. \tag{6}$$

Note that C plays no other role than a typical ratio X/T, and would disappear under a rescaling $T\to CT, V\to V/C$ [13]. More clearly, as in speed relativity, C does not appear in the metrics of the log space, see Appendix B.

The composition law for exponents,

$$V\otimes V'=\frac{V+V'-2\Lambda VV'/C}{1-(\Lambda^2-1)VV'/C^2}, \tag{7}$$

admits two fixed points $C_\pm$ which play an essential role. Starting from an exponent $C_-\leqslant V\leqslant C_+$, one obtains another exponent in the same interval. The two fixed points then characterize the minimum and maximum multifractal exponents of the random field. In Ref. [14], we classified all acceptable transformations according to whether these exponents are zero, finite, or infinite; whether they are equal or different. They characterize the set of so-called log-infinitely divisible laws, including for example the log-normal law ($C_\pm=\pm\infty$). In these preceding papers, we justified the shape of above equations, and showed that it is unique under sound, minimal postulates.

B. Interest of this formalism

The aim of the preceding papers [9,14] was to study the link between successive moments, but at a fixed scale T. Now our aim is to determine how a measurement at one scale is related to another measurement at another scale. This amounts to finding a renormalization trajectory $X(T)$, i.e., the trajectory of a pointlike particle [15] and the dynamical equations which govern it.

To obtain a partial differential equation with respect to the scale, it could in principle be possible to use an extremal action principle based on a Lagrangian formulation. This approach, inspired by Castaing [16], has three objectives: (i) to obtain a new point of view (see Sec. V C); (ii) to have a technique at one's disposal: (see Sec. IV); and (iii) to stimulate new models respecting the scale symmetry, before confronting them to the judgment of experiment.

Of course, this latter objective is interesting only when the physical mechanism under consideration is actually scale invariant, in the sense that coupling between scales ℓ and ℓ' does not explicitly depend separately on them, but only through the combination ℓ/ℓ'. In this case, a Lagrangian which respects the symmetry might become pertinent. Boundary conditions (at lower or higher cutoffs) evidently break the symmetry of solutions, but that does not make the method less powerful.

Of course, the actual expression of the Lagrangian depends on the physical problem under consideration. Symmetry arguments alone cannot entirely determine dynamical equations. But at least they can set strong constraints on their shape, as we will now see by turning to the complete formalism.

IV. LAGRANGIAN FORMALISM

In fact, analogy with special relativity teaches us a lot. In a $(1+1)$-dimensional space-time (x,t), dynamics cannot be coupled to space-time curvature. There are not many possibilities left.

For a free particle in special relativity, the only possible definition of the scalar action is the integral of $-\alpha ds$, and the associated Lagrangian is $\mathcal{L}(x,v)=-\alpha c/\gamma(v)$. Here α must be a positive real number; to ensure correspondence with the Newtonian dynamics, it is identified to mc, where the mass m is a scalar characteristic of the particle.

Now, as far as interactions are concerned, the only possibility left open by symmetry is the existence of an electric field, but no magnetic field. We explore this minimal version of electromagnetism in Sec. IV A.

A. Lagrangian formalism for a scale-invariant process

A free particle corresponds here to a fully scale-invariant process, i.e., with $\dot{X}$ independent of the scale, associated with the Lagrangian, impulsion, and energy (Appendix C):

$$\mathcal{L}=-\frac{MC^2}{\Gamma(\dot{X})},$$

$$\mathcal{P}=\frac{\partial\mathcal{L}(X,\dot{X})}{\partial\dot{X}}, \tag{8}$$

$$\mathcal{E}=\mathcal{P}\dot{X}-\mathcal{L},$$

where M is a constant yet unspecified, and we defined $\Gamma(\dot{X})^{-2}=(\Lambda^2-1)\dot{X}^2/C^2-\Lambda\dot{X}/C+1$ according to Eq. (6).

The corresponding Euler-Lagrange equation is trivial,

$$\frac{d\mathcal{P}}{dT}=0, \tag{9}$$

meaning of course that the exponent $\dot{X}$ does not depend on scale. This validates the present formalism on the strictly self-similar case; however, evidently we do not learn anything as long as we do not specify which physical problem we are studying.

B. Electric field

If we now introduce an electric field (see Sec. III B), the generalized impulsion

$$\mathcal{P}=\partial\mathcal{L}/\partial\dot{X} \tag{10}$$

undergoes a force obeying the Euler-Lagrange equation (D3)

$$MC\frac{d}{dT}\left(\Gamma\left[\Lambda-(\Lambda^2-1)\frac{\dot{X}}{C}\right]\right)=eE, \tag{11}$$

where E is the analog of an electric field; see Appendix D.

Symmetry constraints on the field E, or equivalently on charge distribution, are far less severe than on the tridimensional full electromagnetism [17]. In fact, in such a one-dimensional space, the magnetic field being necessarily zero, Maxwell's equations reduce to

$$\frac{\partial E}{\partial X}=\rho,\qquad \frac{\partial E}{\partial T}=-J. \tag{12}$$

The permittivity is taken equal to 1. Here ρ is the analog of the charge density, and $J^i=(C\rho,J)$ is a bivector with zero divergence:

$$\partial_i J^i=0.$$

Note that E varies smoothly where ρ is finite. Conversely, for a point charge moving along a trajectory $X=R(T)$, then $\rho=e\,\delta[X-R(T)]$ is a Dirac peak, $J=\rho dR/dT$, and $E(X,T)$ is constant piecewise: it is a step function with a discontinuity along the trajectory of the charge.

If eE/MC is given, one can solve Eq. (11) to determine Γ as a function of the log scale T, and then obtain $\dot{X}$. An example relevant to turbulence is provided in our companion paper [18]. To follow this trail further, it is now necessary to propose a model for E, or equivalently for a charge distribution. This is outside the scope of the present paper, since this task relies on experiments and not simply on symmetries considerations. In that respect, it would be interesting to study the ''inverse problem,'' i.e., to determine the charge density required to interpret experimental measurements. If scale-invariance properties are relevant in various fields, such phenomenological approach will eventually lead to a more synthetic theory.

C. Link between different moments

In Refs. [9,14], we studied the link between moments of random fields, at a given scale. In the present paper, we focus on the scale dependence of a given moment. This is described by a Lagrangian theory. Working out the dependence with respect to both scale and order of moments at the same time is feasible, but, in general, complicated. The reason is that the trajectory giving the scale dependence of the moment of order n, say, can be viewed as the trajectory of the moment of order $n-1$, in the ''accelerated'' frame attached to the trajectory of the moment of order 1, a frame which is not Minkowskian in general. Such a computation is formulated by a Euler-Lagrange equation including terms linked to the T dependence of the metrics,

$$\frac{d\dot{X}^{\alpha}}{dS}+\Gamma^{\alpha}_{\beta\mu}\dot{X}^{\beta}\dot{X}^{\mu}=\frac{F^{\alpha}}{M}, \tag{13}$$

where F is the force, S the proper time, and $\Gamma^{\alpha}_{\beta\mu}$ the connection coefficient of the metrics. In the Newtonian limit, however, these connection terms can be simply described by inertial forces. This limit corresponds here to the case of a log-normal statistics for the random process. This case, relevent to hydrodynamical turbulence, is studied in our companion paper [18].

V. CONCLUSION

Our three objectives were to propose a method; to derive predictions *a priori*, as well as tools to analyze *a posteriori* experimental data; and to suggest an interesting point of view. Let us summarize how we have fulfilled these three points.

A. Summary of the method

In mechanics in a space-time with $1+1$ dimensions, you assume that all inertial coordinate systems are equivalent. This is called ''speed relativity.'' You add that space and time are invariant by translation and parity: of course, that does not mean that physical objects are invariant by translation (they are not infinitely long), but that the equations which govern them are. This suffices to tell you what is the composition law for velocities. There is only one free parameter, the square c^2 of the invariant velocities $\pm c$; this is positive to satisfy the causality principle. So you have only two possible mechanics: the Galilean and the Lorentzian kinetics, respectively c finite and infinite, and the associate dynamics, respectively Newtonian and Einsteinian. It is not the symmetry but the experiments which fix the value of c [22].

Since the pioneering work of Nottale [8], approaches inspired by Einstein have been applied to scale symmetry. A principle analogous to speed relativity might impose strong constraints on the possible statistics of random processes in scale-invariant systems [7,9–11]. What is our approach? You consider a random field. You want to know its probability distribution, i.e., a relation between its successive moments; and you also want to know its scale dependence. You render it dimensionless properly, then take its logarithm for convenience, so that you deal with an invariance by translation. Then you assume that you have the free choice of your system of units and subunits. This we call ''exponent relativity.''

As we showed in Ref. [9], this ''exponent relativity'' limits the possible statistics, which can be classified according to the exponents C_+ and C_-. They are not necessarily opposite, since we do not assume that large scales $T\to\infty$ are equivalent to small scales $T\to-\infty$; thus C_+ and C_- can independently be finite or not, so that makes more than just two possible physics.

In the present paper, we showed how to construct an equation to describe how your random field varies with T, i.e., with the scale. The shape of the equation is compatible with exponent relativity, i.e., with the requirements of scale invariance. Of course, the ingredients you plug into this equation, namely, the force source which induces the cascade in scale, or the actual response of your field, need not be scale invariant themselves, but the relation between them is.

B. Summary of the results

We thus derived a Lagrangian formalism, in a rather speculative way, with the following results.

(i) It shows that the analogy between scale symmetry and relativistic mechanics is not only formal, but deeply rooted in their postulates and in their thought processes.

(ii) It introduces a Lagrangian formalism applicable to coupled random fields, suggesting the existence of a quantity conserved along the scales. This provides a framework to generalize usual dimensional analysis.

(iii) As we show in our companion paper [18], this offers a hope to reach a fully scale-covariant description of turbulence which does not give greater importance to large scales or to small scales, generalizing Kolmogorov's approach.

(iv) As we show in our companion paper [18], we can solve the Euler-Lagrange equation in a simple but nontrivial case.

(v) A formalism does not itself make predictions about the force source; however, it is a tool to describe and analyze. Given an experimental result, and keeping turbulence in mind, we know which variable to plot as a function of scale, where we should look for a conserved quantity, and how to prepare a phenomenological model.

C. Summary of our ideas

We are working in the space of the log of the scale and the log of averages of a random field. In this space, a given Lagrangian and its associated deterministic Euler-Lagrange equation define a family of trajectories. Boundary conditions (e.g., fixing a value at two different points, or a value and derivative at one point) select one trajectory in that family unambiguously.

(i) Particular boundary conditions, e.g., rejected at infinity, can select familiar power-law solutions. These are generic solutions, i.e., they are robust and appear independently of the physical mechanism. Of course this appears only in theoretical, idealized cases, such as Kolmogorov's fully developed turbulence (infinite Reynolds number) or critical points in infinite systems.

(ii) However, for other types of boundary conditions, the same equations which respect the same scale symmetry select different solutions, which are also generic in the same sense. Among these are, e.g., stretched exponentials, known to preserve scale symmetry [16,19,18]. In such solutions, boundary conditions are felt over a certain scale range, leading to the natural appearance of a crossover scale visible in experimental data or theoretical predictions. In the example of turbulence, this crossover is the Kolmogorov length, which separates a laminar (''dissipative'') range and an intermittent (''inertial'') range. In critical phenomena, it is the correlation length which separates correlated and uncorrelated scales.

In our opinion there is a flaw in the classical approach, which bases its dimensional analysis only on this crossover scale, considered as the only physically pertinent length scale until it grows larger than the system size. We find hard to reconcile this view with (i), which does not imply any crossover. Conversely, in our point of view both (i) and (ii) fit into a single, coherent picture.

Our method makes no *a priori* assumption about the solution we seek. As long as the underlying physical mechanism is scale invariant, our method is valid, whether cutoffs are close or far away (i.e., whether data are spanned over a small or large number of decades). To analyze experimental data, following Pocheau's recommendations [7], we do not look for slopes of straight lines in a log-log plot, but rather look for physical mechanisms which couple the scales. We thus propose to look for a new variable, which we defined in a formal analogy with an electric field in mechanics (or, equivalently, the analog of a potential; see Ref. [18]). This characterizes the amount of curvature of the log-log plot, and points out the physically relevent length scales of the problem, including of course the boundary conditions. Thus experiment will teach us whether and when this description is pertinent.

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APPENDIX A: ANALOGY WITH GENERAL RELATIVITY

1. Notations

In order to generalize the notion of resolution, let us imagine a continuous slicing of the scale space: at each scale ℓ we have a different subunit, a function of ℓ. This function is continuous; its logarithmic derivative is $\ln K$. This establishes an atlas of overlapping maps covering the scale space. We now generalize n as a continuous variable, namely, the real number T, the differential of which is:

$$dT=\frac{1}{\ln K(\ell)}d\,\ln\left(\frac{\ell}{\ell_0}\right). \tag{A1}$$

Similarly, let ϕ be a random positive physical quantity and ϕ_ℓ the same field defined at scale ℓ. The analogs of the K_i are n independent random fields W_i:

$$\phi_\ell=\mathcal{R}_\ell\times W_1\times\ldots\times W_n, \tag{A2}$$

where $\mathcal{R}_\ell$ is a reference field, possibly random and/or scale dependent [9]. In the analog of the rationalized case, we choose all W_i with an identical probability distribution. This corresponds to infinitely divisible laws [20], with moments following:

$$\left\langle\left(\frac{\phi_\ell}{\mathcal{R}_\ell}\right)^p\right\rangle=\langle W^p\rangle^n=\left(\frac{\ell}{\ell_0}\right)^{\ln\langle W^p\rangle/\ln K}, \tag{A3}$$

where $\langle\ \rangle$ denotes an average on realizations. The more general choice relaxes the requirement of $\langle W_i\rangle$ being identical, assuming only they are greater than 1. An equation in the same spirit as in Eqs. (3) and (A1), could then be written for the random variable ϕ_ℓ, but it would not be very convenient to use, because of the randomness of ϕ_ℓ. To deal only with deterministic numbers [14], we introduce the following p-dependent quantity:

$$\frac{d\,\ln\langle(\phi_\ell/\mathcal{R}_\ell)^p\rangle}{dp}=\frac{\langle\ln(\phi_\ell/\mathcal{R}_\ell)(\phi_\ell/\mathcal{R}_\ell)^p\rangle}{\langle(\phi_\ell/\mathcal{R}_\ell)^p\rangle}, \tag{A4}$$

and thus define the log variable X such as

$$dX=\frac{1}{\ln K'(\ell)}\frac{d^2}{dp}[\ln\langle(\phi_\ell/\mathcal{R}_\ell)^p\rangle], \tag{A5}$$

where K' is a number, related to the moments of the distribution W in a way similar to Eq. (A4).

2. Postulate

Our choice of system of units and subunits is entirely free, so that, in Eqs. (A1) and (A5), K and q can arbitrarily depend, not only on the length scale ℓ, but also on the scale of the field ϕ itself. In analogy with general relativity, we postulate that *physical laws can always be written under a shape suitable to any system of units and subunits*, whether rationalized or not.

This implies that equations must be correctly written according to the tensorial formalism (see Appendix B). The space of the two variables $X^0=CT$ and $X^1=X$ is homogeneous; here C is an exponent characteristic of the physical system under consideration. In this two-dimensional space, the physically significant quantity is the scalar infinitesimal interval

$$dS^2=g_{ij}dX^idX^j, \tag{A6}$$

a quadratic function of the coordinate differentials.

The coefficients g_{ij} depend on the choice of the units and subunits, and may vary with X^i. Physically, they indicate how two different observers can compare their observations, each one using its own set of units and subunits. The variations of the g_{ij}, i.e., their derivative, with respect to the log scale T or to the log X of the measured random field, characterize how rationalized the unit system is.

3. Possible dynamics

Let us mention here the constraints set by the dimension of the space-time on possible dynamics.

(i) The trivial case where the field is not random but deterministic allows only a single variable T, and no dynamics. The same is true for a random quantity involving only one scale, described by a single variable X; there is of course no dynamics.

(ii) A process is a scalar random field, e.g., in isotropic homogeneous turbulence [10]. The variables (T,X) define a space-time with $1+1$ dimensions. Dynamics cannot be coupled to space-time curvature. It is this ''special relativity'' that we develop throughout the present paper.

(iii) With two random fields instead of one, the space-time (CT,X^1,X^2) acquires $1+2$ dimensions. An example is a temperature passively advected by a turbulent isotropic velocity field [21]. In this case, rudimentary versions of gravitation or electromagnetism are allowed.

(iv) With $D\geqslant 3$ different fields, or different components of a vector field, the set of labels $X^i, i=1,\ldots,D$, along with $X^0=CT$, defines an homogeneous space with four dimensions or more, where the tools of general relativity apply.

APPENDIX B: TENSORIAL FORMALISM

The analogy presented in Table I introduces for the (CT,X) space a constant metric tensor $\overline{\overline{g}}$, i.e., all Christoffel symbols being identically zero [17]. The space-time is flat, and we are in special relativity. The only possible metric, left invariant by all similarity transformations (5),

$$\overline{\overline{S^t}}(V).\overline{\overline{g}.\overline{\overline{S}}}(V)=\overline{\overline{g}}, \tag{B1}$$

is, up to a multiplicative constant,

$$\overline{\overline{g}}=(g_{ij})=\begin{pmatrix} 1 & -\Lambda \\ -\Lambda & \Lambda^2-1 \end{pmatrix}; \tag{B2}$$

we keep the solution with determinant -1.

This metric g_{ij} reflects the fact that the relation $g_{ij}R^iR^j=0$, characterizing an intermittent structure, or equivalently $Q\propto \ell^{C_\pm}$, is invariant under any similarity transformation. Its matrix $\overline{\overline{g}}$ is diagonalizable and invertible:

$$\overline{\overline{g}}^{-1}=(g^{ij})=-\begin{pmatrix} \Lambda^2-1 & \Lambda \\ \Lambda & 1 \end{pmatrix}.$$

Note that g_{11} and g^{00} vanish in the log-Poisson case.

More generally, we define a full bidimensional tensorial formalism, also valid in general relativity, i.e., even in a four-dimensional space-time: with the convention that repeated indices are summed as follows: (a) contravariant bivectors, e.g., the infinitesimal radius vector $dX^i(T)=[CdT,dX(T)]$; (b) covariant bivectors, such as $dX_i=g_{ik}dX^k$; (c) a scalar product $dX_idX^i=g_{ik}dX^idX^k=(CdT)^2-2\Lambda dX(CdT)+(\Lambda^2-1)dX^2$; (d) thus a scalar infinitesimal interval $dS=CdT/\Gamma(\dot{X})$, where $\dot{X}=dX(T)/dT$; (e) a scalar derivative d/dS; (f) a covariant derivative $\partial_i=\partial/\partial X^i$; (g) a scalar second-order derivative, the d'Alembertian $\partial_i\partial^i=(1-\Lambda^2)\partial^2_{C^2T^2}-2\Lambda\partial^2_{CT,X}-\partial^2_{X^2}$; and (h) a derived bi-vector $\dot{X}^i=dX^i/dS=[\Gamma(\dot{X}),\Gamma(\dot{X})\dot{X}/C]$ obeying $\dot{X}_i\dot{X}^i=1$.

APPENDIX C: LAGRANGIAN FORMALISM FOR A SCALE-INVARIANT PROCESS

A free particle in special relativity corresponds here to a fully scale-invariant process, i.e., with $\dot{X}$ independent of the scale, associated with the impulsion and the energy:

$$\mathcal{P}=\frac{\partial\mathcal{L}(X,\dot{X})}{\partial\dot{X}}=\frac{\partial}{\partial\dot{X}}\left(-\frac{MC^2}{\Gamma(\dot{X})}\right)=\Gamma MC\left[\Lambda-(\Lambda^2-1)\frac{\dot{X}}{C}\right],$$

$$\mathcal{E}=\mathcal{P}\dot{X}-\mathcal{L}=\Gamma MC^2\left[1-\Lambda\frac{\dot{X}}{C}\right], \tag{C1}$$

In tensorial formalism, this is written

$$\mathcal{P}^i=MC\dot{X}^i=\Gamma MC\begin{pmatrix} 1 \\ \dot{X}/C \end{pmatrix},$$

$$\mathcal{P}_i=\begin{pmatrix} \mathcal{E}/C \\ -\mathcal{P} \end{pmatrix}. \tag{C2}$$

APPENDIX D: LAGRANGIAN FORMALISM WITH AN ELECTRIC FIELD

With a force source, the action, and thus $\Gamma\mathcal{L}$, must still remain a scalar, so that $\mathcal{L}$ can take the form

$$\mathcal{L}=-\frac{1}{\Gamma(\dot{X})}[MC^2+e\dot{X}_iA^i], \tag{D1}$$

where the vector $A^i=(\Phi,A)$. The possible significance of the scalars M and e and the potentials A and Φ depend on the physical quantity we want to describe, e.g., ϵ or δu in turbulence [9].

Since the generalized impulsion is the covariant derivative of the action, we have $\mathcal{P}^i=MC\dot{X}^i+eA^i/C$:

$$\mathcal{P}=\Gamma MC\left[\Lambda-(\Lambda^2-1)\frac{\dot{X}}{C}\right]+\frac{e}{C}[\Lambda\Phi-(\Lambda^2-1)A], \tag{D2}$$

$$\mathcal{E}=\Gamma MC^2\left[1-\Lambda\frac{\dot{X}}{C}\right]+e[\Phi-\Lambda A].$$

The Euler-Lagrange equation yields a force (11):

$$\frac{d}{dT}\left(\Gamma MC\left[\Lambda-(\Lambda^2-1)\frac{\dot{X}}{C}\right]\right)=eE, \tag{D3}$$

where E is the analog of an electric field:

$$E=-\partial_X\Phi-\Lambda\partial_{CT}\Phi+\Lambda\partial_XA+(\Lambda^2-1)\partial_{CT}A. \tag{D4}$$

Let us make three comments:

(i) Note that using the antisymmetric tensor $F_{ik}=\partial_iA_k-\partial_kA_i$, we have the relation $E=F_{01}$, and Eq. (D3) appears as the first component of the vectorial equation

$$C^2\frac{dM\dot{X}_i}{dS}=eF_{ik}\dot{X}^k. \tag{D5}$$

(ii) Whenever the Lagrangian is scale independent, i.e., as long as T does not appear explicitly, the energy $\mathcal{E}$ is equal to the Hamiltonian $\mathcal{H}(\mathcal{P},T)=\mathcal{P}\dot{X}-\mathcal{L}$ and the equation of motion reduces to

$$\Gamma M C^2\left[1-\Lambda\frac{\dot{X}}{C}\right]+e[\Phi-\Lambda A]=\text{const.} \tag{D6}$$

(iii) This formalism has the symmetries of electromagnetism in a one-dimensional space: (a) no magnetic field; (b) invariance by parity; (c) general covariance under similarity transformations [9] (note, e.g., that $E^2=|F|$ is an invariant, i.e., transforms as a scalar, and thus so does E); and (d) electromagnetic gauge invariance, which leaves the choice, e.g., $\Phi=0$, $A=0$ or the more symmetric Lorentz gauge $\partial_i A^i=0$.

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[5] We note X and T our log variables, to avoid confusion with the real space and time coordinates (x,t). What follows also applies to negative quantities, with logarithms taking imaginary values. But this applies only to physical quantities actually *measured*, i.e., labeled as the multiple of a reference unit. This explicitly excludes the use of nonmultiplicative quantities such as relative temperatures, expressed in labels such as celsius or fahrenheits which do not refer to multiplication [3]. A first immediate consequence of this postulate (not a supplementary hypothesis, as is sometimes believed), is that physical quantities must be expressed as products of monoms (power laws) of physical dimensions: e.g., $(\text{mass})\times(\text{length})^2\times(\text{time})^{-1}$ or $(\text{area})^{3/2}\times(\text{time})^{-1}$, is acceptable, while exponentials, sums, or other combinations of monoms are not.

[6] The original meaning of ''gauge invariance'' is invariance of the gauge used to define the unit of a measurement. Weyl coined this term, referring to the possible variations of a measuring rod (or clock period) with respect to the position, and hence its nonintegrability along a path in a curved space; see H. Weyl, *Raum, Zeit, Materie*, 3rd ed. (Teubner, Leipzig, 1920), Chaps. II and IV, p. 242ff, and references therein; Phys. Z. **21**, 649 (1920). This theory was invalidated by experiments, as Weyl himself recognized later [H. Weyl, *Gruppentheorie und Quantenmechanik* (Teubner, Leipzig, 1928)]; W. Pauli, *Theory of Relativity* (Pergamon, London, 1958). However, in quantum mechanics, the same term was used again for a formally analogous problem, namely, the wave equation for an electrically charged particle. Here we come back to the original meaning of ''gauge'' as a measuring rod, but consider its variation with scale and not with position.

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[12] For instance, in other papers [9,14], the physical quantity is the intermittency function $\delta\zeta$; the ratio of resolutions is a constant number, hidden and incorporated in the value of $\delta\zeta(1)$.

[13] In fact, all equations in the text can equivalently be written using only $C_{\pm}$, e.g.,

$$\begin{pmatrix}T'\\X'\end{pmatrix}=\Gamma(V)\begin{pmatrix}1-V(1/C_{+}+1/C_{-}) & V/C_{+}C_{-}\\ -V & 1\end{pmatrix}\begin{pmatrix}T\\X\end{pmatrix}.$$

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[15] Pocheau [7] showed that, for a certain class of problems, all scales must be renormalized together, and that the interaction between scales is nonlocal. This does not apply to isotropic turbulence, but could apply, e.g., to turbulent front analysis. This situation is similar to quantum relativistic theory, and would necessarily involve backwards running time, i.e., values of K or $\langle q\rangle$ equal to or less than 1, as in mechanics; see, e.g., the necessity for antiparticles in R. Feynman, *Elementary Particles and the Laws of Physics* (Cambridge University Press, Cambridge, 1987).

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